APOCALYPSE OF MATHEMATICS IN 2021

This paper provides one solution to the division of an angle into three equal parts. It also provides an angle division into *n* equal parts using one caliper and one ruler in the finite number of procedures.

This paper is delivered via the Internet ; It is thus made available worldwide for anyone interested in this solution.

The paper is presented in Serbian and English language. Its final review and publishing procedures will be carried out by joint involvement of mathematical associations of the U.S.A., the Russian Federation, and the People's Republic of China.

I apologize for not being able to render a high-quality translation into Russian and Chinese languages.

In case of a favorable outcome for this paper, the U.S.A, the Russian Federation, the People's Republic of China, and the Internet will be highly rewarded.

In the paper, I use some theorems and assumptions for clarification purposes.

1) Each circular arc $\widehat{AB}$ on circle $k(o,r)$ has a corresponding central angle α and peripheral angle $\frac{\alpha}{2}$. Figure 1.

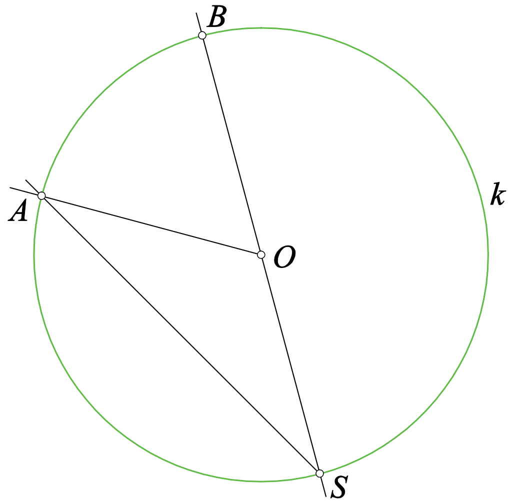
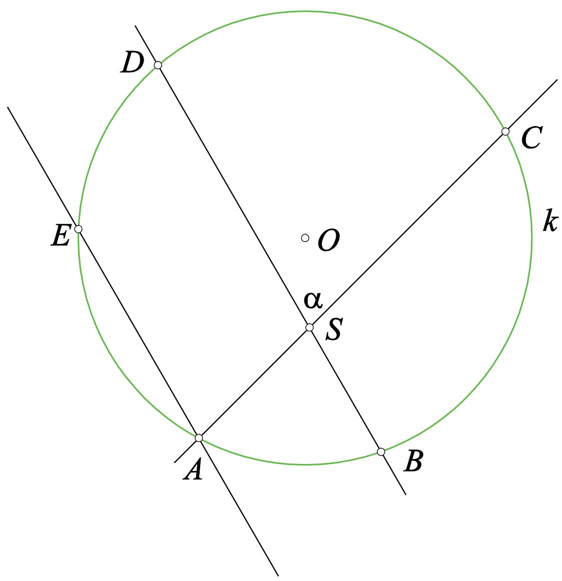


Figure 1

2) If the vertex of angle α is inside circle $k(o,r)$, the two circular arcs $\widehat{AB}$ and $\widehat{CD}$ belong to angle α. Figure 2. The sum of these arcs is equal to arc $\widehat{EC}$ if point A is positioned on circle k and if points A, S and C are collinear, with straight-line AE being parallel to straight-line BD.



$$\widehat{AB} + \widehat{CD} = \widehat{EC}$$

Figure 2

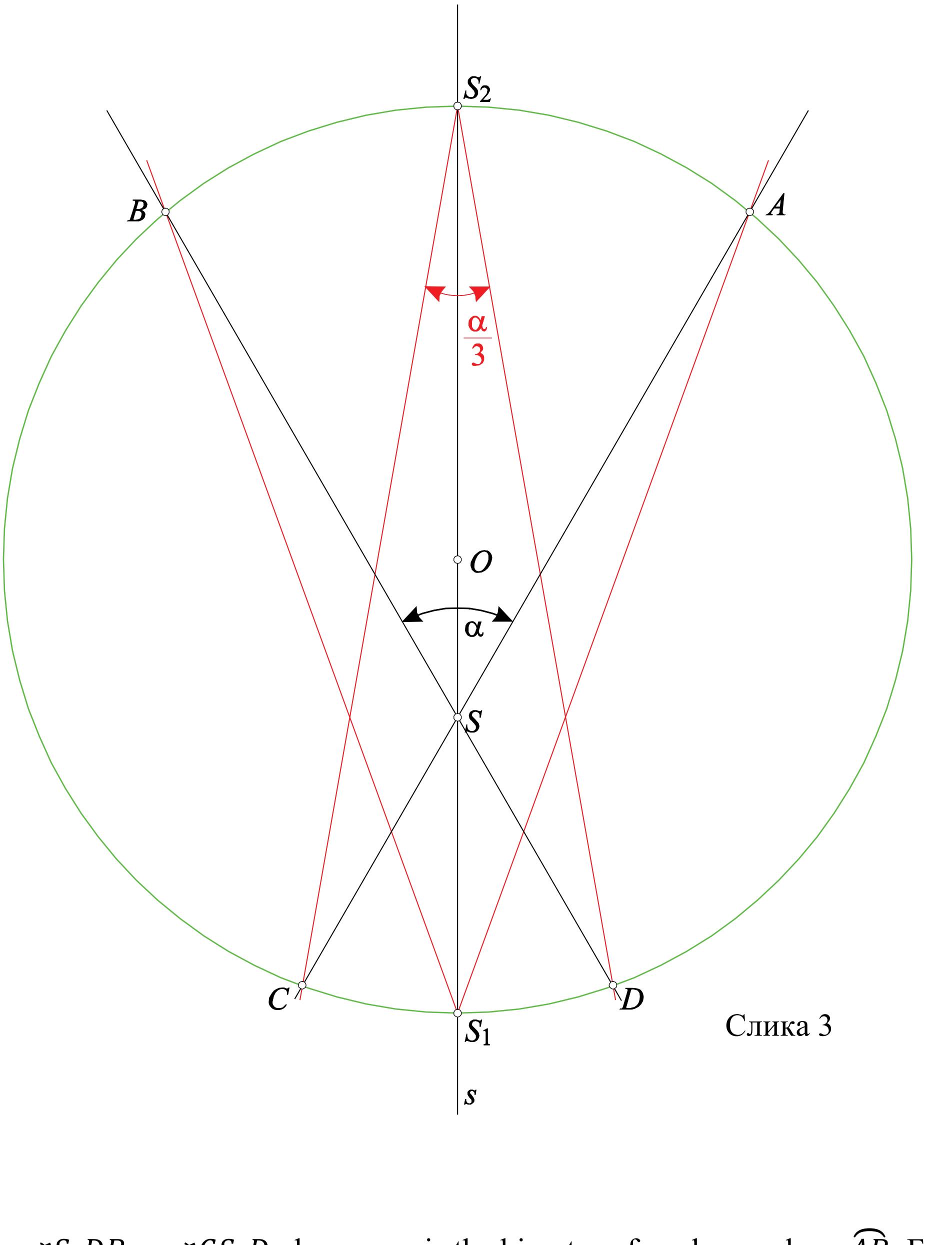
The last equation is true, as arcs $\widehat{AB}$ *and* $\widehat{ED}$ are the equal arcs made by parallel chords of the same circle.

I

DIVISION OF ANGLE α INTO THREE EQUAL SECTIONS

For the purpose of this proof, we will make one assumption to prove theorem 1:

Assumption 1: If the vertex of angle α is inside circle $k(o,r)$ and if arc $\overset{\frown}{AB}$ is two times bigger than cross-arc $\overset{\frown}{CD}$, angle α is then divided into three equal sections. Figure 3.



Слика 3

$\sphericalangle S_2CA = \sphericalangle S_2DB = \sphericalangle CS_2D$, because s is the bisector of angle α and arc $\overset{\frown}{AB}$. Finally:

$$\sphericalangle AS_1B = 2 \cdot \sphericalangle CS_2D = \frac{2\alpha}{3}$$

Theorem 1: Central angle $\alpha = \sphericalangle AO_2B$ is given in circle $k_2(O_2, O_2A)$; Points *A* and *B* are given on circle k_2. Figure 4.

May the straight-line *s* be the bisector of angle α and arc $\widehat{AB}$, as well as arc $\widehat{IH}$ $(I \in k_2 \wedge H \in k_2)$ whereas the latter is equal to half the angle $\widehat{AB}$.

May straight-lines i and h be the straight-lines containing points I and H and being parallel to axis *s*. Then straight-lines i and h intersect the extensions of the arms of angle α through vertex O_2 and in points *I'* and *H'* to make equation $\widehat{IH} = \widehat{I'H'}$.

Points *I, H, I'* and *H'* belong to circle $k_3(O_3, O_3I)$. (Point O_3 is the intersection between bisector *s* and straight line *IH'*).

Let's design $k_3(O_3, O_3I)$.

Its intersections with the arms of angle α are designated A_3 and B_3. Axis *s* intersects circle $k_3(O_3, O_3I)$ in points S_1 and S_3. It also intersects circle k_2 in point S_2.

Based on the conditions indicated hereinbefore, we argue as follows: Circles k_2 and k_3 have common chord *IH,* but also equal chords as follows:

$AS_2 = A_3S_3$

<u>Proof:</u> When presenting Figure 4, the equality of the following chords was taken into account:

$AS_2 = BS_2 = IH = I'H'$.

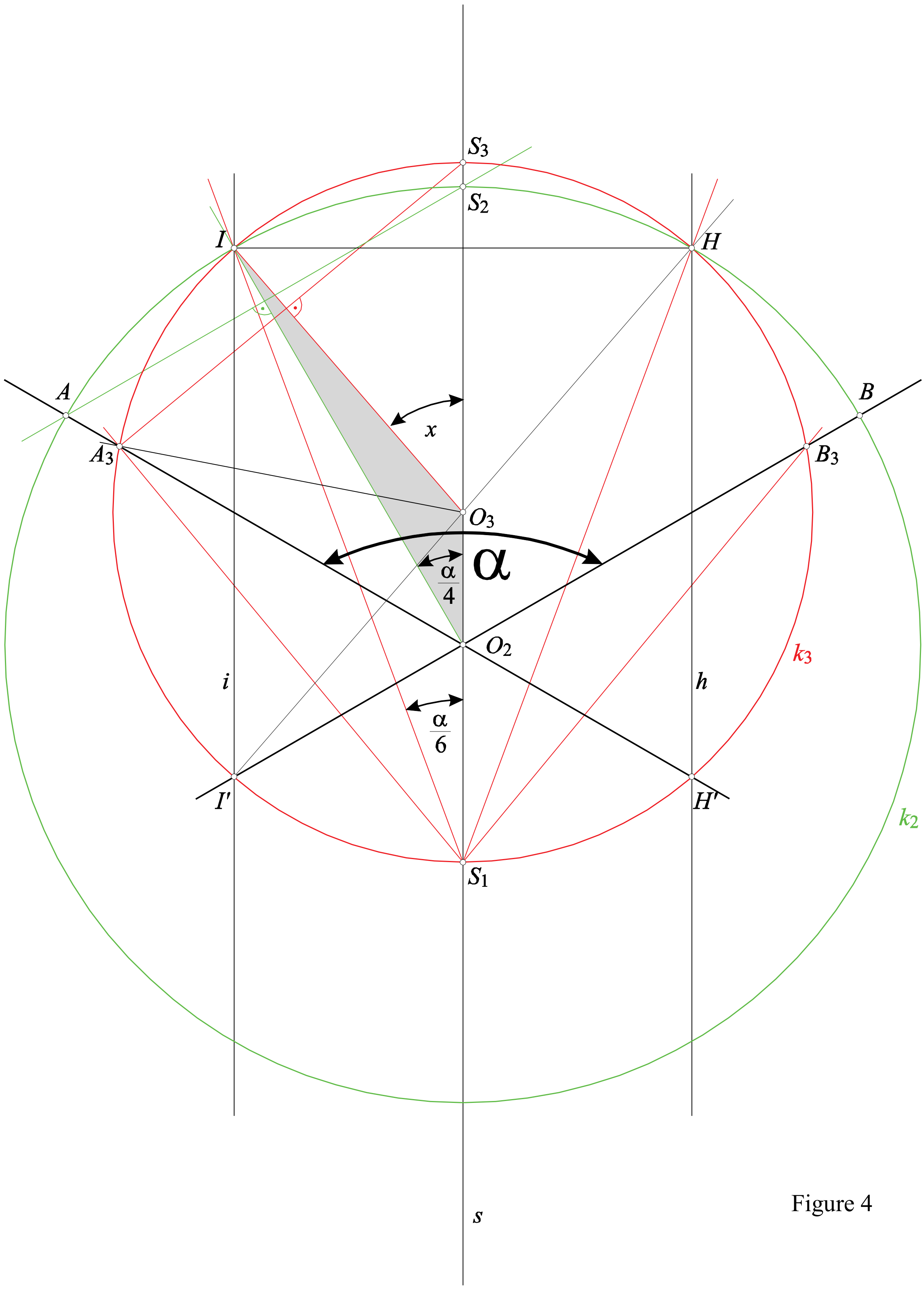


Figure 4

Let’s observe triangle AO_2S_2 on circle k_2 and this circle’s radius O_2I which is right in reference to chord AS_2 . Therefore:

$$AS_2 = 2 \cdot r_2 \sin\frac{\alpha}{4} \quad (1)$$

From triangle $A_3O_3S_3$ to circle k_3 , straight-line O_3I is right in reference to chord A_3S_3. Therefore:

$$A_3S_3 = 2 \cdot r_3 \sin x \quad (2), \quad \sphericalangle x = \sphericalangle IO_3S_3$$

The application of the Sinus Theorem to triangle IO_2O_3 gives the result as follows:

$$r_2 : r_3 = \sin x : \sin\frac{\alpha}{4} \quad (3), \quad \text{thus:}$$

$$r_3 = \frac{r_2 \cdot \sin\frac{\alpha}{4}}{\sin x} \quad (4)$$

If r_3 from the relation (4) is included in the relation (2), the result is as follows:

$$A_3S_3 = 2 \cdot \frac{r_2 \sin\frac{\alpha}{4}}{\sin x} \cdot \sin x = 2r_2 \sin\frac{\alpha}{4} \quad \text{Congratulations!}$$

This value is equal to the value of relation (1), therefore:

$$AS_2 = A_3S_3$$

Theorem 1 is hereby proven.

On circle k_3 , angle α has two corresponding arcs $\overset{\frown}{A_3B_3}$ and $\overset{\frown}{I'H'}$. Therefore:

$\overset{\frown}{A_3B_3} = 2\,\overset{\frown}{I'H'}$ so we can finally conclude that:

$$\sphericalangle IS_1H = \frac{\alpha}{3} \text{ and } \sphericalangle A_3S_1B_3 = 2\frac{\alpha}{3}, \qquad \sphericalangle x = \frac{\alpha}{3}$$

(**Note**: For the first time in the mathematical history of mankind, the relation (3) provides a method to either calculate or construct angle $\frac{\alpha}{3}$ using angle $\frac{\alpha}{4}$, arbitral line $O_2I = r_2$ and – based on this arbitral line - the constructed line $O_3I = r_3$, with no ocurrence of transcendent numbers.)

Now, assumption (1) and theorem (1) are considered the analysis of the angle division into three equal sections. Based on these, we can conceive the design of angle $\frac{\alpha}{3}$ if angle α is known.

<u>Example 1:</u> Divide the given angle α into three equal parts.

1) Analysis: We used assumption (1) and theorem (1) to conduct the analysis. Based on the obtained concept, we make the following design:

2) Design: An angle $\alpha = \sphericalangle aO_2b$ is given in the plane. Let's design bisector s for angle α and circle $k_2(O_2, OA)$, where A is an arbitral point positioned on arm a of angle α. Figure 5.

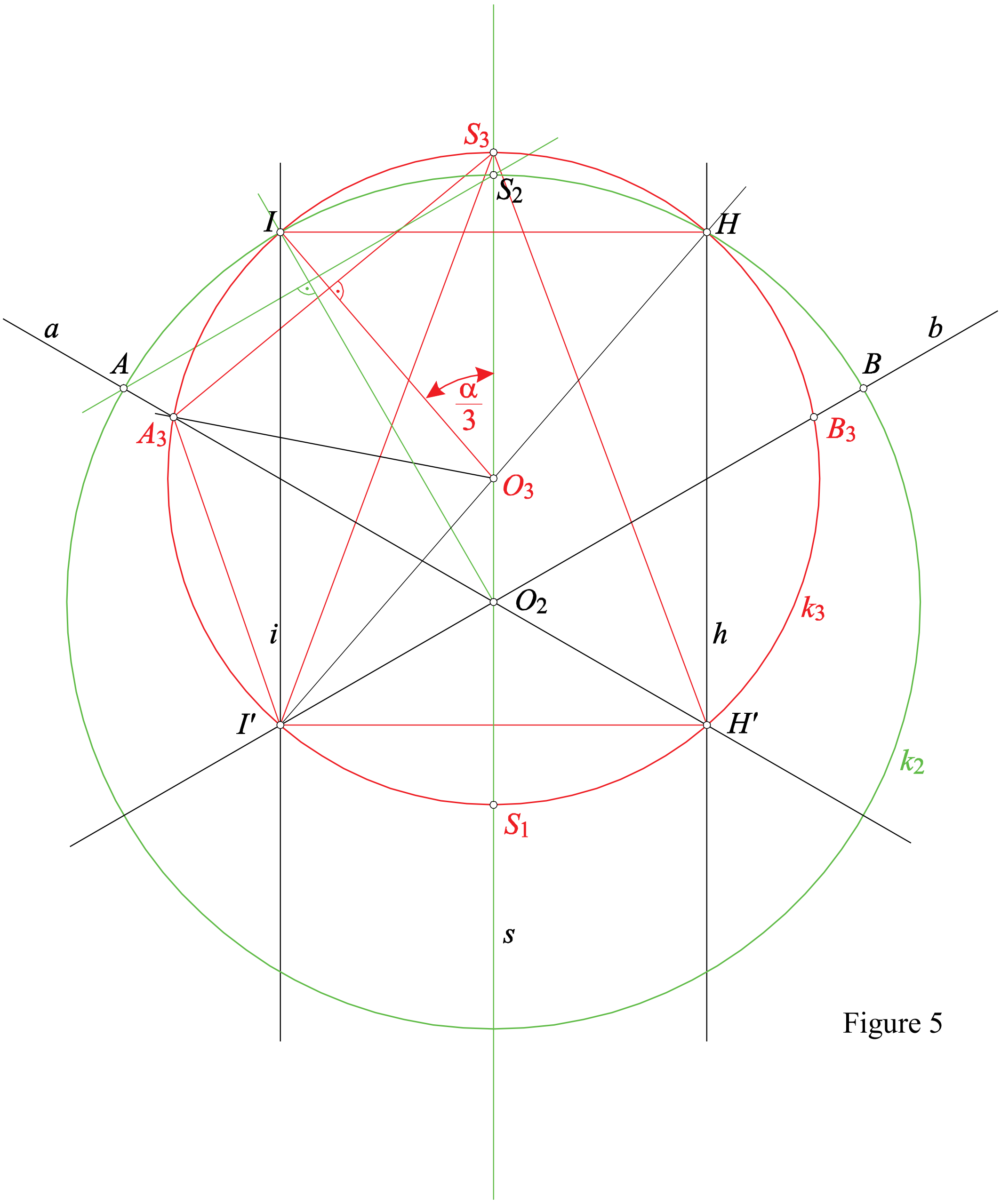


Figure 5

When bisector s intersects circle k_2 in point S_2, then arc $\widehat{AS_2} = \widehat{BS_2}$. As the analysis showed that points I and H are positioned in the middle of arcs $\widehat{AS_2}$ and $\widehat{BS_2}$. Then the following shall apply for circle k_2:

$\widehat{AS_2} = \widehat{BS_2} = \widehat{IH}$.

From points I and H , let's design straight-lines i and h parallel to axis s and intersecting the extensions of the arms of angle α in points I' and H'. Design now the straight-line $I'H$. According to the analysis, this line's intersection with straight-line s determines point O_3. Design circle k_3. Circle k_3 intersects axis s in points S_1 and S_3 , and the arms of angle α in points A_3 and B_3.

Let's design chords A_3S_3 and B_3S_3 , as well as chords $I'H'$ and AS_2.

Based on theorem (1) , $AS_2 = A_3S_3 = IH = I'H'$ i

$$\sphericalangle A_3H'S_3 = \sphericalangle B_3I'S_3 = \sphericalangle I'S_3H' = \frac{\alpha}{3}$$

thus making the design procedure complete.

3) Proof. Theorem (1) and the design prove that peripheral angles are:

$$\sphericalangle I'S_3H' = \frac{\alpha}{3} \wedge \sphericalangle A_3H'S_3 = \sphericalangle B_3I'S_3 = \frac{\alpha}{3}$$

4) Discussion: The task has a unique solution if angle $\alpha \leq \frac{2\pi}{3}$. If $\alpha > \frac{2\pi}{3}$, arcs on circle k_3 will overlap, which may be misleading.

For the entire design process, we used only one caliper and one ruler in the finite number of procedures.

Consequences and Prospective Applications of this Work

- Based on this angle division into three equal sections, consecutive application of this procedure produces angles such as

$$\frac{\alpha}{3^k}, \qquad k \in N$$

and division as follows:

$$\frac{\alpha}{3^k 2^r}, \qquad k, r \in N$$

- If the angle of 360° or its parts are divided, multiple polygons might be designed.

II

DIVISION OF ANGLE α INTO $\frac{\alpha}{2n+1}$ EQUAL PARTS IF THE $\frac{\alpha}{2n}$ DIVISION IS KNOWN

Assumption 2: If the given circle $k(o,r)$ contains angle $\alpha = \sphericalangle AO_2B$ divided into two parts and such arcs $\widehat{AB}$ and $\widehat{CD}$ that $\widehat{AB} = 2n \cdot \widehat{CD}$, then circle *k* contains angle

$\sphericalangle CS_3D = \dfrac{\alpha}{2n+1}$ Figure 6.

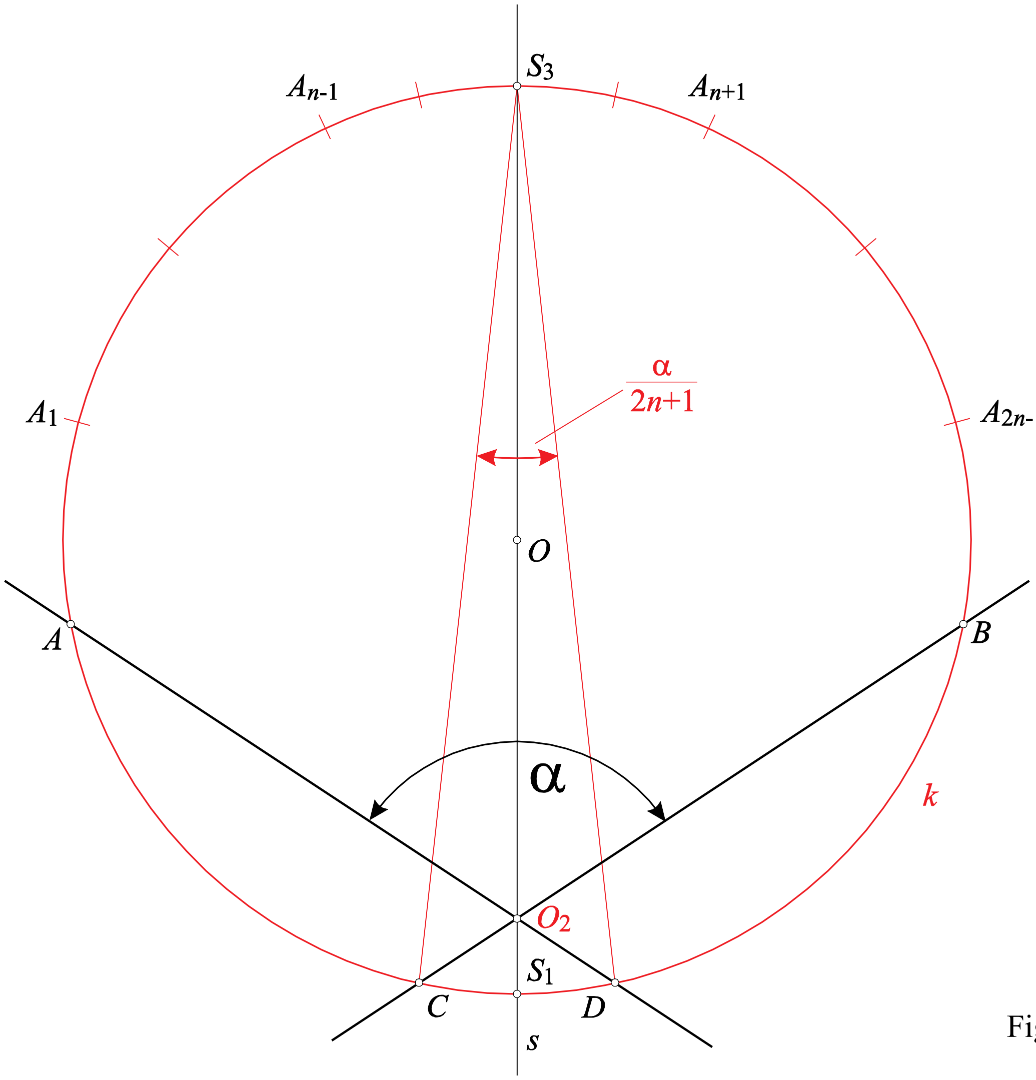


Figure 6

Theorem 2:

An angle $\alpha = \sphericalangle AO_2B$ is given. It is divided on arc $\widehat{AB}$ by points $A, A_1 \ldots A_{n-1}, S_2, A_{n+1}, \ldots A_{2n-1}, B$ into $2n$ equal parts. May points *I* and *H* be positioned on circle k_{2n} between points A_{n-1} and S_2 , as well as between points S_2 and A_{n+1} so that points *I* and *H* are located in the middle of these parts (arcs). May it also be as follows:

$\widehat{IH} = \widehat{A_{n-1}S_2} = \widehat{S_2A_{n+1}}.$

May straight-line s be the bisector of angle α and bisector of arc $\widehat{IH}$. May straight-line s intersect circle $k_{2n}(O_2 O_2 A)$ in point S_2.

May straight-lines i i h be such that $I \in i$ and $H \in h$ that are parallel to axis s. May straight-lines i and h intersect the extensions of the arms of angle α through vertex O_2 in points I' and H'. Now, points I, H, I' and H' are the four points of the same circle k_{2n+1} whose center is point O_3 that is located in the intersection of bisector s and straight-line $I'H$, while the radius of circle k_{2n+1} is $r_{2n+1} = O_3 I$. Let's design circle k_{2n+1}. This circle and circle k_{2n} have common points I and H and chord IH. Circle k_{2n+1} intersects the arms of angle α in points A' and B'. This circle intersects axis s in points S_1 and S_3. Figure 7.

If we connect point O_2 with points $A_1, A_2 \dots A_{n-1}, I$, we will get the radii of circle k_{2n} intersecting circle k_{2n+1} in points $A'_1, A''_1, A'_2, A''_2, \dots A'_{n-1}, A''{}_{n-1}, I \; and \; I''$.

The points marked $'$ are located on the k_{2n+1} circle segment spread out from point A' to S_3. The points marked $''$ are located on the k_{2n+1} circle segment spread out from point S_1 to point H'. These are not designated due to the lack of space. In point O_2, the angle between these radii is $\frac{\alpha}{2n}$. Figure 7.

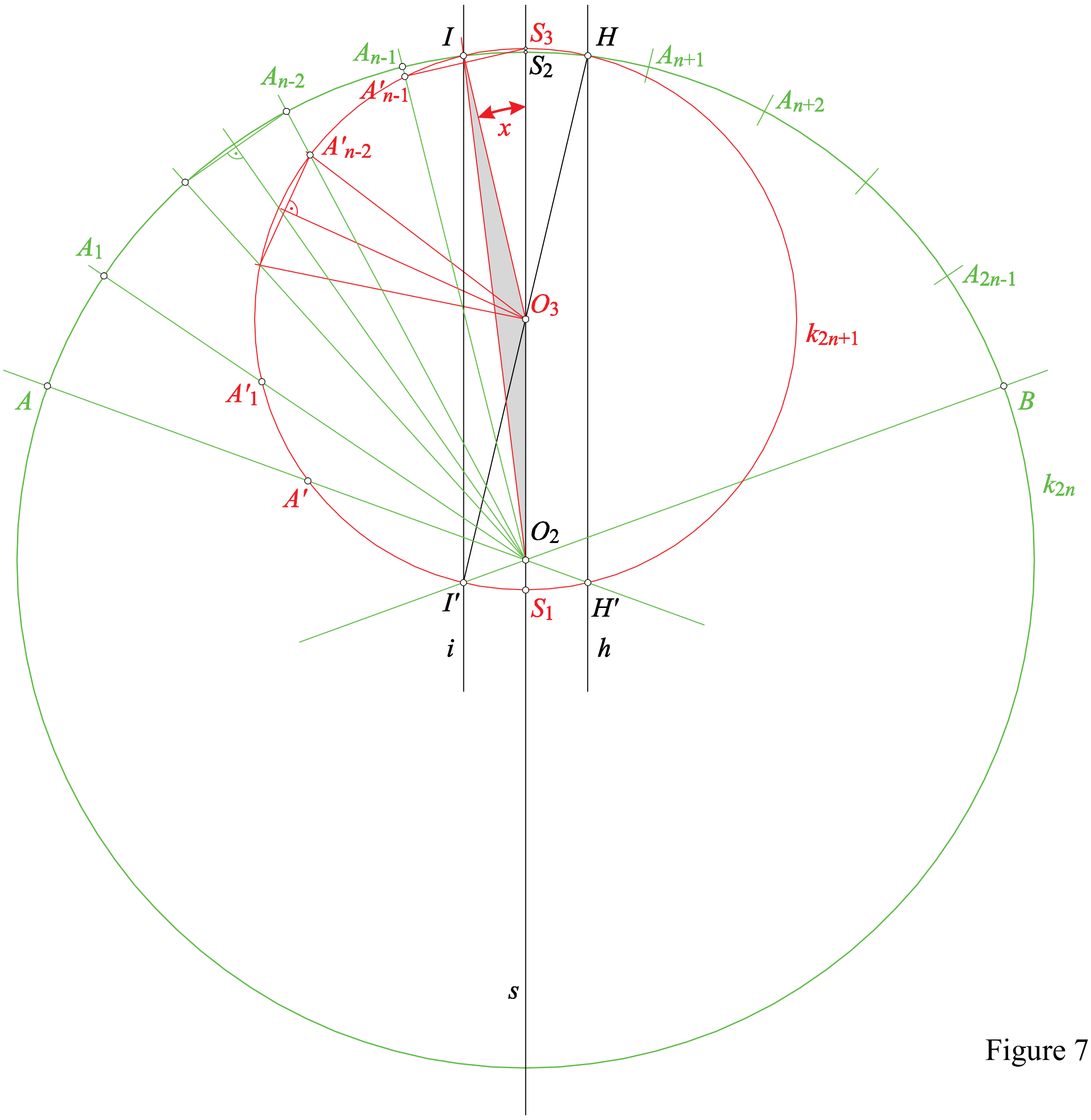


Figure 7

Theorem 2 - Argument. Chords $A'A'_1, A'_1A'_2, \ldots\ A'_{n-1}S_3$ - mutually constructed by the aforementioned radii O_2A_1, O_2A_2 ... on circle k_{2n+1} – equally amount to *IH*.

The angle formed from point O_2 with each chord amounts to $\frac{\alpha}{2n}$, whereas the peripheral angle on circle k_{2n+1} amounts to $\frac{\alpha}{2n+1} = \sphericalangle x$ for each chord.

<u>Proof:</u> *n* equal chords *IH* belong to arc $\widehat{AS_2}$ in reference to central angle $\sphericalangle AO_2S_2 = \frac{\alpha}{2}$ of circle k_{2n}.

Also, $\left(n + \frac{1}{2}\right)$ equal chords *IH* belong to arc $\widehat{A'S_3}$ in reference to angle $\sphericalangle \mathrm{A}'O_2S_3 = \frac{\alpha}{2}$ of circle k_{2n+1}. The expression $\left(n + \frac{1}{2}\right)$ refers to arc $\widehat{A'S_3} = n \cdot \left(\widehat{IH}\right)$, while arc $\widehat{S_1H'} = \frac{1}{2}\left(\widehat{IH}\right)$. The expressions: $n \cdot \frac{\alpha}{2n}$ and $\frac{\alpha}{2n+1} \cdot \left(n + \frac{1}{2}\right)$ are equal for they correspond to angle $\frac{\alpha}{2}$.

It means that $n \cdot \frac{\alpha}{2n} = \frac{\alpha}{2n+1} \cdot \left(n + \frac{1}{2}\right)$ becomes an identity with multiplication of the equation by $2n \cdot (2n+1)$.

Although the equality of chords of k_{2n} and k_{2n+1} has been presented, let's behold Figure 9 and the chords $A_{n-1}S_2$ i $A'_{n-1}S_3$. Based on the above analysis:

$$O_2A_{n-1} = r_{2n}, \qquad O_3I = r_{2n+1}, \qquad \sphericalangle IO_3S_3 = \sphericalangle x \ \ i \ \ \sphericalangle IO_2O_3 = \frac{\alpha}{4n}$$

The application of the Sinus Theorem for triangle IO_2O_3 gives the results as follows:

1) $$r_{2n} : r_{2n+1} = \sin x : \sin\frac{\alpha}{4n} \Rightarrow r_{2n+1} = \frac{r_{2n} \cdot \sin\frac{\alpha}{4n}}{\sin x}$$

Chords $A_{n-1}S_2$ i $A'_{n-1}S_3$ amount to:

2) $$A_{n-1}S_2 = 2 \cdot r_{2n} \cdot \sin\frac{\alpha}{4n}$$

3) $$A'_{n-1}S_3 = 2 \cdot r_{2n+1} \cdot \sin x$$

If the value from the relation (1): $r_{2n+1} = \frac{r_{2n} \cdot \sin\frac{\alpha}{4n}}{\sin x}$

is included in the relation (3), the result is as follows:

$$A'_{n-1}S_3 = 2 \cdot \frac{r_{2n} \cdot \sin\frac{\alpha}{4n}}{\sin x} \cdot \sin x = 2r_{2n}\sin\frac{\alpha}{4n} = A_{n-1}S_2$$

We have proven the equality of chords of k_{2n} and k_{2n+1}.

In relation (1):

$$r_{2n}: r_{2n+1} = \sin\frac{\alpha}{2n+1} : \sin\frac{\alpha}{2n}$$

we can easily find one value if we know other three values.

The relation between angle $\frac{\alpha}{2n}$, angle $\frac{\alpha}{2n+1}$, r_{2n} and r_{2n+1} is established for the first time.

Example 1. For the given angle α , design the angle $\frac{\alpha}{17}$.

1) Analysis: We used assumption (2) and theorem (2) to conduct the analysis. Based on these, we can conceive the design of angle $\frac{\alpha}{17} = \frac{\alpha}{2 \cdot 8+1}$ if angle α is known.

2) Design: An angle $\alpha = \sphericalangle AO_2B$ is given in the plane. Dividing the angle α and its parts into halves we obtain the angle of $\frac{\alpha}{16}$. Figure 8.

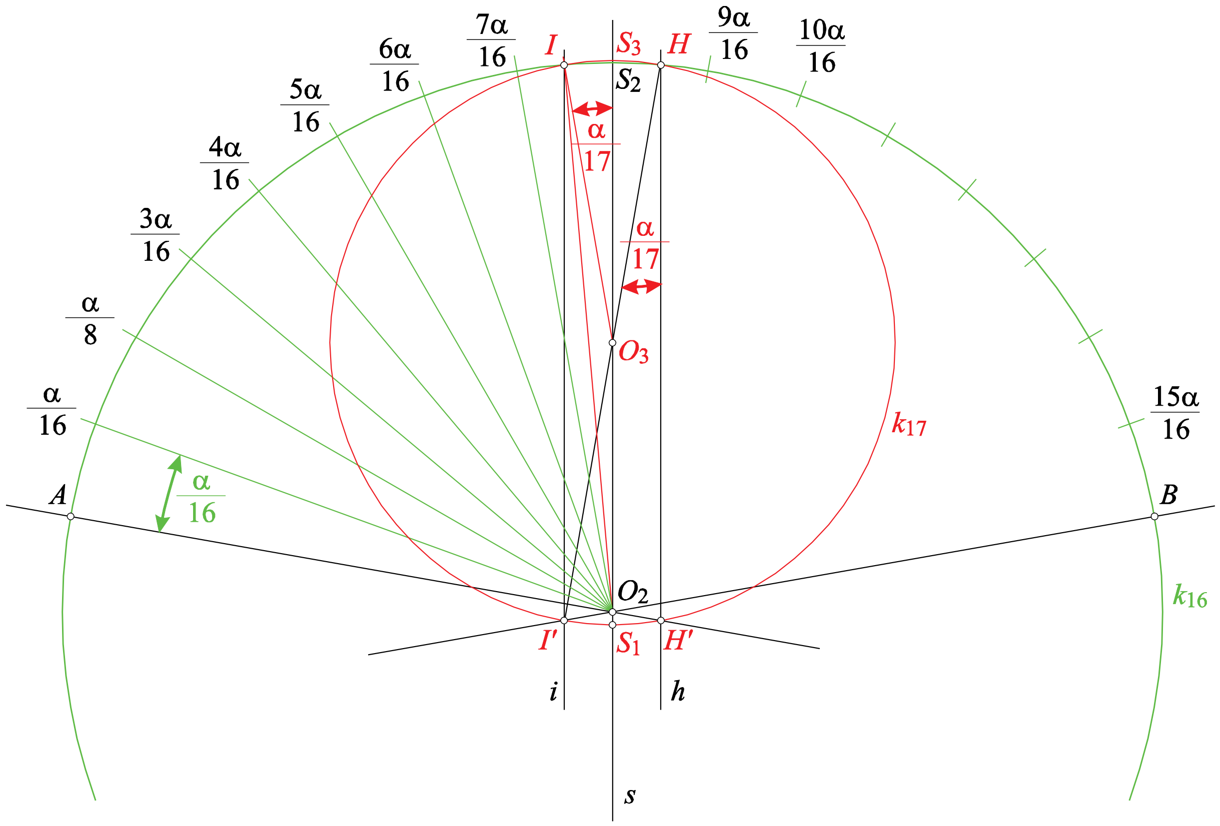


Figure 8

Using the design procedure based on theorem 2, we determine the points $I, H, I' i\, H'$. Design circle $k_{17}(O_3, O_3 I)$. Angle $I'HH' = \frac{\alpha}{17}$.

3) Proof: Based on assumption 2 and theorem 2, angle $\sphericalangle I'HH' = \frac{\alpha}{17}$.

4) Discussion: The task has an unique solution if $\alpha + \frac{\alpha}{17} \leq 180°$

(**Note 2**: Figure 7 shows circle $k(O_2, O_2 A)$ which is divided into $2n$ equal arcs of arc $\widehat{AB}$. When one of these arcs is brought into the IH position, straight-line s is its bisector. When straight lines i and h are designed to be parallel to axis s $(I \in i,\ H \in h)$and intersecting the extensions of the arms of angle α through vertex O_2 in points I' and H' , then the assumption points to 2 $\sphericalangle II'H = \sphericalangle I'HH' = \frac{\alpha}{2n+1}$. We do not have to support this proof with either theorem 2 or the design of the circle k_{2n+1}.)

Application: If we divide the angle of $\alpha = \frac{2\pi}{3} = 120°$ into 17 equal parts, we would get that: $3 \cdot \frac{\alpha}{17} = \frac{360°}{17}$, which is the central angle of the proper 17-angle easy to be immediately designed.

This method of division of angle α into n equal parts gives the opportunity to design any proper polygon whose central angle amounts to $3 \cdot \frac{120°}{n}$.

The difficulties occuring when, for example, $n = 35$ are solved in the following manner: $35 = 5 \cdot 7$. First, the angle a is divided into 5 (4 + 1) equal parts, then one of these parts is divided in the 3rd step of the procedure into 7 (6 + 1) equal sections, after it was previously divided into 3 (2 + 1) sections. In the second step, we use a half of the angle to take the 3rd step for we know $\frac{\alpha}{2}$ and therefore we construct $\frac{\alpha}{5 \cdot 7}$. We solve other situations in a similar way as well. Also, we can construct angle $\frac{\alpha}{35}$ immediately after the $\frac{\alpha}{17}$ angle is constructed; we will take a half of the angle $\frac{\alpha}{17}$ to obtain $\frac{\alpha}{34}$, and then construct the $\frac{\alpha}{35}$ angle as described above.

This design of proper polygons may have significant applications in algebra, such as graphic solutions to $x^n - 1 = 0,\ n \in N$ equations.

As for unit circle with radius $r = 1$, all solutions to the equation $x^n - 1 = 0$ are located on the circle. The first solutions is as follows:

$$Z_1 = \cos\frac{2\pi}{n} + i\ \sin\frac{2\pi}{n}$$

Each subsequent solution is obtained by means of the multiplication of the previous one by Z_1. This procedure can also be used for the $x^n - 1$ polynom factorization.

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